ON p-ADIC q-l-FUNCTIONS AND SUMS OF POWERS

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Abstract. In this paper, we give an explicit p-adic expansion of

$$\sum_{\substack{j=1\\(j,p)=1}}^{np} \frac{(-1)^j}{[j]_q^r}$$

as a power series in n. The coefficients are values of p-adic q-l-function for q-Euler numbers.

§1. Introduction

Let p be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, cf.[1, 4, 6, 10]. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Kubota and

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Leopoldt proved the existence of meromorphic functions, $L_p(s,\chi)$, defined over the p-adic number field, that serve as p-adic equivalents of the Dirichlet L-series, cf.[10, 11]. These p-adic L-functions interpolate the values

$$L_p(1-n,\chi)=-\frac{1}{n}(1-\chi_n(p)p^{n-1})B_{n,\chi_n},\ \text{for } n\in\mathbb{N}=\{1,2,\cdots,\}\ ,$$

where $B_{n,\chi}$ denote the nth generalized Bernoulli numbers associated with the primitive Dirichlet character χ, and $\chi_n=\chi w^{-n}$, with w the *Teichmüller* character, cf.[8, 10]. In [10], L. C. Washington have proved the below interesting formula:

$$\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{1}{j^r}=-\sum_{k=1}^{\infty}\binom{-r}{k}(pn)^kL_p(r+k,w^{1-k-r}),\ \text{where } \binom{-r}{k} \text{ is binomial coefficient.}$$

To give the q-extension of the above Washington result, author derived the sums of powers of consecutive q-integers as follows:

$$(*)\qquad \sum_{l=0}^{n-1}q^l[l]_q^{m-1}=\frac{1}{m}\sum_{l=0}^{m-1}\binom{m}{l}q^{ml}\beta_l[n]_q^{m-l}+\frac{1}{m}(q^{mn}-1)\beta_m,\ \text{see [6, 7]}\ ,$$

where β_m are q-Bernoulli numbers. By using (*), we gave an explicit p-adic expansion

$$\begin{aligned}\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{q^j}{[j]_q^r}&=-\sum_{k=1}^{\infty}\binom{-r}{k}[pn]_q^kL_{p,q}(r+k,w^{1-r-k})\\&-(q-1)\sum_{k=1}^{\infty}\binom{-r}{k}[pn]_q^kT_{p,q}(r+k,w^{1-r-k})-(q-1)\sum_{a=1}^{p-1}B_{p,q}^{(n)}(r,a:F),\end{aligned}$$

where $L_{p,q}(s,\chi)$ is p-adic q-L-function (see [7]). Indeed, this is a q-extension result due to Washington, corresponding to the case $q=1$, see [10]. For a fixed positive integer d with $(p,d)=1$, set

$$\begin{aligned}&X=X_d=\varprojlim_N \mathbb{Z}/dp^N,\\&X_1=\mathbb{Z}_p, X^*=\bigcup_{\substack{0<a<dp\\(a,p)=1}}a+dp\mathbb{Z}_p,\\&a+dp^N\mathbb{Z}_p=\{x\in X|x\equiv a\pmod{p^N}\},\end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$, (cf.[3, 4, 9]). We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x,y) \to (a,a)$, cf.[3]. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \text{ cf.[1, 3, 4, 7, 8, 9]},$$

which represents a q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit of those sums(as $n \to \infty$) if this limit exists. The q-Volkenborn integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

(1)

$$I_q(f) = \int_X f(x)d\mu_q(x) = \int_{X_d} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x)q^x, \text{ cf. [2, 3] }.$$

It is well known that the familiar Euler polynomials $E_n(z)$ are defined by means of the following generating function:

$$F(z,t) = \frac{2}{e^t+1}e^{zt} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!}, \text{ cf.[1, 5]}.$$

We note that, by substituting $z = 0$, $E_n(0) = E_n$ are the familiar n-th Euler numbers. Over five decades ago, Carlitz defined q-extension of Euler numbers and polynomials, cf.[1, 4, 5]. Recently, author gave another construction of q-Euler numbers and polynomials (see [1, 5, 9]). By using author's q-Euler numbers and polynomials, we gave the alternating sums of powers of consecutive q-integers as follows:

$$2\sum_{l=0}^{n-1}(-1)^l[l]_q^m = (-1)^{n+1}\sum_{l=0}^{m-1}\binom{m}{l}q^{nl}E_{l,q}[n]_q^{m-l} + \left((-1)^{n+1}q^{nm}+1\right)E_{m,q},$$

where $E_{l,q}$ are q-Euler numbers (see [5]). From this result, we can study the p-adic interpolating function for q-Euler numbers and sums of powers due to author [7]. Throughout this paper, we use the below notation:

$$[x]_q = \frac{1-q^x}{1-q} = 1 + q + q^2 + \cdots + q^{x-1},$$
$$[x]_{-q} = \frac{1-(-q)^x}{1-q} = 1 - q + q^2 - q^3 + \cdots + (-q)^{x-1}, \text{ cf.[5, 9]}.$$

Note that when p is prime $[p]_q$ is an irreducible polynomial in $Q[q]$. Furthermore, this means that $Q[q]/[p]_q$ is a field and consequently rational functions $r(q)/s(q)$ are well defined $\mod [p]_q$ if $(r(q), s(q)) = 1$. In a recent paper [5] the author constructed the new q-extensions of Euler numbers and polynomials. In Section 2, we introduce the q-extension of Euler numbers and polynomials. In Section 3 we construct a new q-extension of Dirichlet's type l-function which interpolates the q-extension of generalized Euler numbers attached to χ at negative integers. The values of this function at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with at negative integers. The purpose of this paper is to construct the new q-extension of generalized Euler numbers attached to χ due to author and prove the existence of a specific p-adic interpolating function which interpolate the q-extension of generalized Bernoulli polynomials at negative integer. Finally, we give an explicit p-adic expansion

$$\sum_{\substack{j=1\\(j,p)=1}}^{np} \frac{(-1)^j}{[j]_q^r},$$

as a power series in n. The coefficients are values of p-adic q-l-function for q-Euler numbers.

2. Preliminaries

For any non-negative integer m, the q-Euler numbers, $E_{m,q}$, were represented by

$$\frac{2}{[2]_q}\int_{\mathbb{Z}_p} q^{-x}[x]_q^m d\mu_{-q}(x) = E_{m,q} = 2\left(\frac{1}{1-q}\right)^m \sum_{i=0}^{m}\binom{m}{i}(-1)^i \frac{1}{1+q^i}, \text{ see [9] .} \tag{2}$$

Note that $\lim_{q\to 1} E_{m,q} = E_m$. From Eq.(2), we can derive the below generating function:

$$F_q(t) = 2e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{1}{1+q^j}(-1)^j(\frac{1}{1-q})^j \frac{t^j}{j!} = \sum_{j=0}^{\infty} E_{n,q}\frac{t^n}{n!}. \tag{3}$$

By using p-adic q-integral, we can also consider the q-Euler polynomials, $E_{n,q}(x)$, as follows:

$$E_{n,q}(x) = \frac{2}{[2]_q}\int_{\mathbb{Z}_p} q^{-t}[x+t]_q^n d\mu_{-q}(t) = 2(\frac{1}{1-q})^n \sum_{k=0}^{n}\binom{n}{k}\frac{(-q^x)^k}{1+q^k}, \text{ see [5, 9].} \tag{4}$$

Note that

$$E_{n,q}(x) = \frac{2}{[2]_q} \int_{\mathbb{Z}_p} ([x]_q + q^x [t]_q)^n q^{-t} d\mu_{-q}(x) = \sum_{j=0}^{n} \binom{n}{j} q^{jx} E_{j,q} [x]_q^{n-j}. \tag{5}$$

By (4), we easily see that

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = F_q(x,t) = 2e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{(-1)^j}{1+q^j} q^{jx} (\frac{1}{1-q})^j \frac{t^j}{j!}. \tag{6}$$

From (6), we derive

$$F_q(x,t) = 2\sum_{n=0}^{\infty} (-1)^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{7}$$

3. On the q-analogue of Hurwitz's type ζ-function associated with q-Euler numbers

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. It is easy to see that

$$E_{n,q}(x) = [m]_q^n \sum_{a=0}^{m-1} (-1)^a E_{n,q^m}(\frac{a+x}{m}), \text{ see [1, 5] },$$

where m is odd positive integer. From (7), we can easily derive the below formula:

$$E_{k,q}(x) = \frac{d^k}{dt^k} F_q(x,t)|_{t=0} = 2\sum_{n=0}^{\infty} (-1)^n [n+x]_q^k. \tag{8}$$

Thus, we can consider a q-ζ-function which interpolates q-Euler numbers at negative integer as follows:

Definition 1. *For $s \in \mathbb{C}$, define*

$$\zeta_{E,q}(s,x) = [2]_q \sum_{m=1}^{\infty} \frac{(-1)^n}{[n+x]_q^s}.$$

Note that $\zeta_{E,q}(s,x)$ is meromorphic function in whole complex plane.

By using Definition 1 and Eq.(8), we obtain the following:

Proposition 2. *For any positive integer k, we have*

$$\zeta_{E,q}(-k,x)=E_{k,q}(x).$$

Let χ be the Dirichlet character with conductor $f\in\mathbb{N}$. Then we define the generalized q-Euler numbers attached to χ as

$$F_{q,\chi}(t)=2\sum_{n=0}^{\infty}e^{[n]_q t}\chi(n)(-1)^n=\sum_{n=0}^{\infty}E_{n,\chi,q}\frac{t^n}{n!}. \tag{9}$$

Note that

$$E_{n,\chi,q}=[f]_q^n\sum_{a=0}^{f-1}\chi(a)(-1)^aE_{n,q^f}(\frac{a}{f}),\ \text{where } f(=odd)\in\mathbb{N}\ . \tag{10}$$

By (9), we easily see that

$$\frac{d^k}{dt^k}F_{q,\chi}(t)|_{t=0}=E_{k,\chi,q}=2\sum_{n=1}^{\infty}\chi(n)(-1)^n[n]_q^k \tag{11}$$

Definition 3. *For $s\in\mathbb{C}$, we define Dirichlet's type l-function as follows:*

$$l_q(s,\chi)=[2]_q\sum_{n=1}^{\infty}\frac{\chi(n)(-1)^n}{[n]_q^s}.$$

From (11) and Definition 3, we can derive the below theorem.

Theorem 4. *For $k\geq 1$, we have*

$$l_q(-k,\chi)=E_{k,\chi,q}.$$

In [5], it was known that

$$2\sum_{l=0}^{n-1}(-1)^l[l]_q^m=\left((-1)^{n+1}q^nE_{m,q}(n)+E_{m,q}\right),\ \text{where } m,n\in\mathbb{N}. \tag{12}$$

From (4) and (12), we derive

$$\begin{aligned} &2\sum_{l=0}^{n-1}(-1)^l[l]_q^m \\ &= (-1)^{n+1}\sum_{l=0}^{m-1}\binom{m}{l}q^{nl}E_{l,q}[n]_q^{m-l} + \left((-1)^{n+1}q^{nm}+1\right)E_{m,q}. \end{aligned} \tag{13}$$

Let s be a complex variable, and let a and $F(= odd)$ be the integers with $0 < a < F$. We now consider the partial q-zeta function as follows:

$$H_q(s, a : F) = \sum_{\substack{m\equiv a(F)\\ m>0}} \frac{(-1)^m}{[m]_q^s} = (-1)^a\frac{[F]_q^{-s}}{2}\zeta_{E,q^F}(s, \frac{a}{F}). \tag{14}$$

For $n \in \mathbb{N}$, we note that $H_q(-n, a : F) = (-1)^a\frac{[F]_q^n}{2}E_{n,q^F}(\frac{a}{F})$. Let χ be the Dirichlet's character with conductor $F(= odd)$. Then we have

$$l_q(s, \chi) = 2\sum_{a=1}^{F}\chi(a)H_q(s, a : F). \tag{15}$$

The function $H_q(s, a : F)$ will be called the q-extension of partial zeta function which interpolates q-Euler polynomials at negative integers. The values of $l_q(s, \chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with $l_q(s, \chi)$ at the negative integers in Section 4.

§4. p-ADIC q-l-FUNCTIONS AND SUMS OF POWERS

We define $< x >=< x : q >= \frac{[x]_q}{w(x)}$, where $w(x)$ is the *Teichmüller* character. When $F(= odd)$ is multiple of p and $(a, p) = 1$, we define a p-adic analogue of (14) as follows:

$$H_{p,q}(s, a : F) = \frac{(-1)^a}{2} < a >^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F}, \text{ for } s \in \mathbb{Z}_p. \tag{16}$$

Thus, we note that

(17)

$$H_{p,q}(-n,a:F)=\frac{(-1)^a}{2}<a>^n\sum_{j=0}^{n}\binom{n}{j}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^jE_{j,q^F}$$
$$=\frac{(-1)^a}{2}w^{-n}(a)[F]_q^nE_{n,q^F}(\frac{a}{F})=w^{-n}(a)H_q(-n,a:F),\text{ for }n\in\mathbb{N}.$$

We now construct the p-adic analytic function which interpolates q-Euler number at negative integer as follows:

$$l_{p,q}(s,\chi)=2\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)H_{p,q}(s,a:F). \tag{18}$$

In [5, 9], it was known that

$$E_{k,\chi,q}=\frac{2}{[2]_{q^f}}\int_X\chi(x)[x]_q^kq^{-x}d\mu_{-q}(x),\text{ for }k\in\mathbb{N}.$$

For $f(=odd)\in\mathbb{N}$, we note that

$$E_{n,\chi,q}=[f]_q^n\sum_{a=0}^{f-1}\chi(a)(-1)^aE_{n,q^f}(\frac{a}{f}).$$

Thus, we have

(18-1)

$$l_{p,q}(-n,\chi)=2\sum_{\substack{a=1\\(p,a)=1}}^{F}\chi(a)H_{p,q}(-n,a:F)=\frac{2}{[2]_{q^f}}\int_{X^*}\chi w^{-n}(x)[x]_q^nq^{-x}d\mu_{-q}(x)$$
$$=E_{n,\chi w^{-n},q}-[p]_q^n\chi w^{-n}(p)E_{n,\chi w^{-n},q^p}.$$

In fact,

$$l_{p,q}(s,\chi)=2\sum_{a=1}^{F}(-1)^a<a>^{-s}\chi(a)\sum_{j=0}^{\infty}\binom{-s}{j}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^jE_{j,q^F},\text{ for }s\in\mathbb{Z}_p. \tag{19}$$

This is a p-adic analytic function and has the following properties for $\chi = w^t$:

$$l_{p,q}(-n, w^t) = E_{n,q} - [p]_q^n E_{n,q^p}, \text{ where } n \equiv t \ (\mod p-1), \tag{20}$$

$$l_{p,q}(s,t) \in \mathbb{Z}_p \text{ for all } s \in \mathbb{Z}_p \text{ when } t \equiv 0(\mod p-1). \tag{21}$$

If $t \equiv 0(\mod p-1)$, then $l_{p,q}(s_1, w^t) \equiv l_{p,q}(s_2, w^t)(\mod p)$ for all $s_1, s_2 \in \mathbb{Z}_p$, $l_{p,q}(k, w^t) \equiv l_{p,q}(k+p, w^t)(\mod p)$. It is easy to see that

$$\frac{1}{r+k-1}\binom{-r}{k}\binom{1-r-k}{j} = \frac{-1}{j+k}\binom{-r}{k+j-1}\binom{k+j}{j}, \tag{22}$$

for all positive integers r, j, k with $j, k \geq 0$, $j+k > 0$, and $r \neq 1-k$. Thus, we note that

$$\frac{1}{r+k-1}\binom{-r}{k}\binom{1-r-k}{j} = \frac{1}{r-1}\binom{-r+1}{k+j}\binom{k+j}{j}. \tag{22-1}$$

From (22) and (22-1), we derive

$$\frac{r}{r+k}\binom{-r-1}{k}\binom{-r-k}{j} = \binom{-r}{k+j}\binom{k+j}{j}. \tag{23}$$

By using (13), we see that

$$\begin{aligned}
\sum_{l=0}^{n-1}\frac{(-1)^{Fl+a}}{[Fl+a]_q^r} &= \sum_{l=0}^{n-1}(-1)^l(-1)^a\left([a]_q + q^a[F]_q[l]_{q^F}\right)^{-r} \\
&= -\sum_{s=0}^{\infty}[a]_q^{-r}\left(\frac{[F]_q}{[a]_q}\right)^s q^{as}(-1)^a\binom{-r}{s}\frac{(-1)^n}{2}\sum_{l=0}^{s-1}\binom{s}{l}q^{nFl}E_{l,q^F}[n]_{q^F}^{s-l} \\
&\quad -\sum_{s=0}^{\infty}[a]_q^{-r}\left(\frac{[F]_q}{[a]_q}\right)^s q^{as}(-1)^a\binom{-r}{s}\frac{((-q^{Fs})^n-1)}{2}E_{s,q^F}.
\end{aligned} \tag{24}$$

For $s \in \mathbb{Z}_p$, we define the below T-Euler polynomials:

$$T_{n,q}(s, a:F) = (-1)^a < a >^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}[\frac{a}{F}]_{q^F}^{-k}q^{ak}\left((-1)^n q^{nFk} - 1\right)E_{k,q^F}. \tag{25}$$

Note that $\lim_{q\to 1} T_{n,q}(s,a:F)=0$, if n is even positive integer. From (23) and (24), we derive

$$\begin{aligned}&\sum_{l=0}^{n-1}\frac{(-1)^{Fl+a}}{[Fl+a]_q^r}\\ &=-\sum_{s=0}^{\infty}\binom{-r}{s}[a]_q^{-r}\left(\frac{[F]_q}{[a]_q}\right)^s\frac{(-q^s)^a(-1)^n}{2}\sum_{l=0}^{s-1}\binom{s}{l}q^{nFl}E_{l,q^F}[n]_{q^F}^{s-l}\\ &-\frac{w^{-r}(a)}{2}T_{n,q}(r,a:F).\end{aligned} \tag{26}$$

First, we evaluate the right side of Eq.(26) as follows:
(27)

$$\begin{aligned}&\sum_{s=0}^{\infty}\binom{-r}{s}[a]_q^{-r}\left(\frac{[F]_q}{[a]_q}\right)^s\frac{(-q^s)^a(-1)^n}{2}\sum_{l=0}^{s-1}\binom{s}{l}q^{nFl}E_{l,q^F}[n]_{q^F}^{s-l}\\ &=\sum_{k=0}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}[a]_q^{-k-r}q^{ak}(-1)^n[Fn]_q^k\frac{(-1)^a}{2}\sum_{l=0}^{\infty}\binom{-r-k}{l}q^{al}\left(\frac{[F]_q}{[a]_q}\right)^l E_{l,q^F}.\end{aligned}$$

It is easy to check that

$$q^{nFl}=\sum_{j=0}^{l}\binom{l}{j}[nF]_q^j(q-1)^j=1+\sum_{j=1}^{l}\binom{l}{j}[nF]_q^j(q-1)^j. \tag{28}$$

Let
(29)

$$K_{p,q}(s,a:F)=\frac{(-1)^a}{2}<a>^{-s}\sum_{l=0}^{\infty}\binom{-s}{l}q^{al}\left(\frac{[F]_q}{[a]_q}\right)^l E_{l,q^F}\sum_{j=1}^{l}\binom{l}{j}[nF]_q^j(q-1)^j.$$

Note that $\lim_{q\to 1} K_{p,q}(s,a;F)=0$. For $F=p$, $r\in\mathbb{N}$, we see that

$$2\sum_{a=1}^{p-1}\sum_{l=0}^{n-1}\frac{(-1)^{a+pl}}{[a+pl]_q^r}=2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{[j]_q^r}. \tag{30}$$

For $s\in\mathbb{Z}_p$, we define p-adic analytically continued function on $\mathbb{Z}_p$ as

$$\begin{aligned}K_{p,q}(s,\chi)&=2\sum_{a=1}^{p-1}\chi(a)K_{p,q}(s,a:F),\\ T_{p,q}(s,\chi)&=2\sum_{a=1}^{p-1}\chi(a)T_{n,q}(s,a:F),\ \text{where } k,n\geq 1\ .\end{aligned} \tag{31}$$

From (24)-(31), we derive

$$2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{[j]_q^r}=-\sum_{k=0}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k l_{p,q}(r+k,w^{-r-k})$$
$$-\sum_{k=0}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k K_{p,q}(r+k,w^{-r-k})-T_{p,q}(r,w^{-r}).$$

Therefore we obtain the following theorem:

Theorem 5. *Let p be an odd prime and let $n\geq 1$, and $r\geq 1$ be integers. Then we have*

$$\begin{aligned}2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{[j]_q^r}&=-\sum_{k=0}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k l_{p,q}(r+k,w^{-r-k})\\&-\sum_{k=0}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k K_{p,q}(r+k,w^{-r-k})-T_{p,q}(r,w^{-r}).\end{aligned}\tag{32}$$

For $q=1$ in (32), we have

$$2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{j^r}=-\sum_{k=0}^{\infty}\frac{r}{k+r}\binom{-r-1}{k}(-1)^n(pn)^k l_p(r+k,w^{-r-k}),$$

where n is positive even integer.

Remark. *Let p be an odd prime. Then we have*

$$\sum_{j=1}^{p-1}\frac{(-1)^j q^j}{[j]_q}=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}.$$

Proof. To prove Remark, it is sufficient to show that

$$\begin{aligned}\left(\sum_{j+1}^{p-1}\frac{(-1)^j}{[j]_q}\right)^2 &= \left(\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\right)\left(\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q} - (1-q)\sum_{j=1}^{p-1}(-1)^j\right)\\ &= \left(\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\right)\left(\sum_{j=1}^{p-1}(-1)^j\left(\frac{1}{[j]_q} - (1-q)\right)\right)\\ &= \left(\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\right)\left(\sum_{j=1}^{p-1}\frac{(-1)^j q^j}{[j]_q}\right).\end{aligned}$$

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